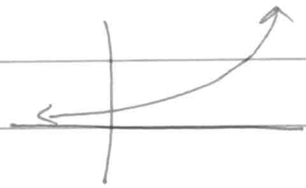
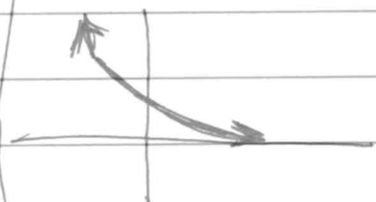
1060 - Day 26
Fall 2013

5.4 - Exponential Growth & Decay

setup. Mathematics gives a powerful language for discussing [redacted] the world around us

(5) Two of the most common math models are

Name	Picture	Equation
exponential Growth		$y = P_0 \cdot e^{k\cancel{x}}$ k is positive #
exponential Decay		$y = P_0 \cdot e^{k\cancel{x}}$ k is negative #

why is this?

eg: $y = e^{-2x} = (e^{-2})^x = \left(\frac{1}{e^2}\right)^x$

& $\frac{1}{e^2} < 1$

What things do exp. growth & decay model?

Eg: Total Population ~~when~~ if people have children at a constant rate

Disease spread if people infect others at a constant rate

Radioactive Decay: if decays @ constant ~~neg~~ rate

In general:

if rate of growth is proportional to size

that is:

if $\frac{\text{rate of growth}}{\text{size}} = k$ is constant

To see why this works, we'd need calculus.
So we'll focus on examples.

Before we go into modeling defn's, lets see what it does for us

Eg: Suppose a disease spreads so that

$$I(\boxed{t}) = \#\text{ ill after } \boxed{t} \text{ days} = 10 \cdot e^{\ln(2) \cdot \boxed{t}}$$

[what does this do for us?]

(a) Notice $\quad I(\boxed{0}) = 10 \cdot e^{\ln(2) \cdot \boxed{0}} = 10 \cdot e^{0}$

$$I(0) = I_0 = 10$$

10

$\uparrow$ # ill at start of outbreak.

(b) compute # ill after 3 days

$$I(\boxed{3}) = 10 \cdot e^{\ln(2) \cdot \boxed{3}}$$

$$= 10 \cdot \left(e^{\ln(2)}\right)^3$$

$$= 10 (2)^3$$

$$= 10 \cdot 8$$

$\boxed{I(3) = 80}$ = # ill after 3 days.

(c) when does "# of ill" equal 300 ?

know

$$I(t) = 10 \cdot e^{\ln(2) \cdot t}$$

WANT t s.t.

$$300 = 10\, e^{\ln(2) \cdot \boxed{t}}$$

solve for $\boxed{t}$

10-15

$$30 = e^{\ln(2) \cdot t}$$

$$\ln(30) = \ln(2) \cdot t$$

$$t = \frac{\ln(30)}{\ln(2)} \approx 4.907 \text{ days}$$

using calculator

$\boxed{\text{the \# ill} = 300 \text{ after} \approx 4.9 \text{ days}}$

Sometimes these models clean up a bit

$$\boxed{I(t) = 10 \cdot e^{\ln(2) t}}$$

$$= 10 \cdot (e^{\ln(2)})^t$$

both equal

$$\boxed{I(t) = 10(2)^t}$$

How do you find a model like this?

ALL exponential Growth & Decay models have the form

$$P(t) = P_0 \cdot e^{(k \cdot t)}$$

growth constant

initial population

Use the given information to find in P_0 & k

Eg:

Suppose a disease outbreak begins with 7 people and the number of ill doubles after 2 hours

(a) find an expression for # of ill after t hours.

Know: $P(t) = P_0 e^{kt} = 7 \cdot e^{kt}$

P_0 = # ill @ start (0 hours) = 7

Know: $P(\boxed{2}) = 14 = 7 \cdot e^{k \cdot \boxed{2}}$

So

$$14 = 7 \cdot e^{k \cdot \boxed{2}}$$

init pop

double init pop

solve for k

$$2 = e^{k\cdot\boxed{2}}$$

$$\ln(2) = k\cdot 2$$

$$k = \frac{\ln(2)}{2}$$

So

$$P(\boxed{t}) = 7\cdot e^{\frac{\ln(2)}{2}\boxed{t}}$$

(b) predict # of ill after 4 Hours

$$P(\boxed{4}) = 7\cdot e^{\frac{\ln(2)}{2}\cdot\boxed{4}}$$

$$= 7\cdot e^{2\cdot\ln(2)}$$

$$= 7\cdot\left(e^{\ln(2)}\right)^2$$

$$= 7\cdot(2)^2$$

$$= 7\cdot 4$$

$$P(4) = 28$$

(c) when will ~~pop~~ # ill reach 7000?

find t s.t.

$$P(t) = 7000$$

find t s.t.

$$7000 = 7 \cdot e^{\frac{\ln(2)}{2} t}$$

solve for t.

$$1000 = e^{\frac{\ln(2)}{2} t}$$

$$\ln(1000) = \frac{\ln(2)}{2} t$$

$$t = \frac{\ln(1000)}{\ln(2)} \cdot 2 \approx 19.9 \text{ hours}$$

(d) How long till population triples?

find t s.t.

$$3 \cdot 7 = 7 \cdot e^{\frac{\ln(2)}{2} t}$$

$$3 = e^{\frac{\ln(2)}{2} t}$$

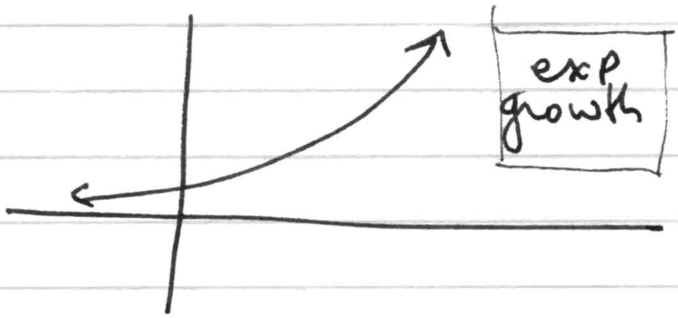
$$\ln(3) = \frac{\ln(2)}{2} t$$

$$t = \frac{2 \cdot \ln(3)}{\ln(2)} \approx 3.17 \text{ hours...}$$

In exponential growth,

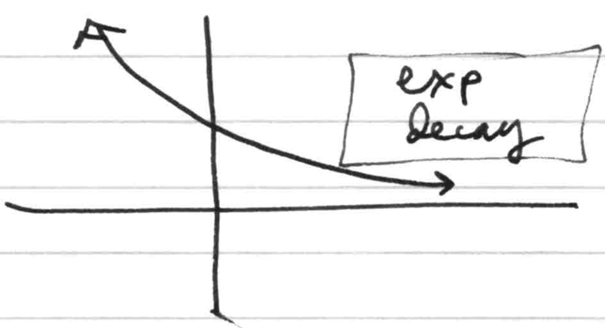
the population doubles (or triples or ...)

at a constant rate.



In exponential decay,
the population halves (or thirds or...)
at a constant rate.

<u>Half life</u> of radioactive material
= amount of time to cut size in <u>half</u>.



Eg: A certain radioactive substance has a half-life of 7 years.

If you start with 100 grams,

(a) find an expression for mass after t years

$P_0 = 100$ g.

$P(\boxed{t}) = 100 \cdot e^{k\boxed{t}}$

after 7 years, cut weight in half

$50 = P(\boxed{7}) = 100 \cdot e^{k\boxed{7}}$

~~solve for k~~

$50 = 100 \, e^{7k}$

$\frac{1}{2} = e^{7k}$

Both are OK

$\ln\left(\frac{1}{2}\right) = 7k$

$k = \frac{\ln\left(\frac{1}{2}\right)}{7} = \frac{\ln(2^{-1})}{7} = \frac{-\ln(2)}{7}$

So

$P(t) = 100 \, e^{\frac{-\ln(2)}{7} t}$

(b) find mass after 14 years

$$P(\boxed{14}) = 100 \cdot e^{\frac{-\ln(2)}{7} \cdot \boxed{14}}$$

OR

$$P(\boxed{14}) = 100 \cdot e^{\frac{\ln(\frac{1}{2})}{7} \cdot \boxed{14}}$$

$$= 100 \cdot e^{\ln(\frac{1}{2}) \cdot 2}$$

$$= 100 \cdot \left(e^{\ln(\frac{1}{2})}\right)^2$$

$$= 100 \cdot \left(\frac{1}{2}\right)^2$$

$$= 100 \cdot \frac{1}{4}$$

$$\boxed{P(14) = 25}$$

Full Solutions to HW problems 1-3.

(1) $P_0 = 2000$

Doubles in 3 hours $\Rightarrow$ $P(3) = 2 \cdot 2000 = 4000$

exponential growth

$$\Rightarrow P(t) = P_0 \cdot e^{kt}$$

$$P(t) = 2000 \cdot e^{kt}$$

Know

$$P(\boxed{3}) = \underbrace{4000 = 2000 \cdot e^{k\boxed{3}}}$$

solve for k

$$4000 = 2000 \, e^{k \cdot 3}$$

$$2 = e^{k \cdot 3}$$

$$\ln(2) = \ln(e^{(k \cdot 3)})$$

$$\ln(2) = k \cdot 3$$

$$k = \frac{\ln(2)}{3}$$

(a) $$P(t) = 2000 \cdot e^{\left(\left(\frac{\ln(2)}{3}\right) \cdot t\right)}$$

(b) $P(6) = 2000 \cdot e^{\left(\frac{\ln(2)}{3}\right) \cdot 6}$

$= 2000 \cdot e^{\ln(2) \cdot \frac{6}{3}}$

$= 2000 \cdot e^{2 \cdot \ln(2)} = 2000 \cdot \left(e^{\ln(2)}\right)^2$

$= 2000 \cdot (2)^2 = 8000$

(c) when is $P(t) = 22{,}000$?

$22{,}000 = 2{,}000 \cdot e^{\frac{\ln(2)}{3} \cdot t}$

$11 = e^{\frac{\ln(2)}{3} \cdot t}$

$\ln(11) = \frac{\ln(2)}{3} \cdot t$

$\boxed{t = \frac{3 \cdot \ln(11)}{\ln(2)}}$

(2) Divides in 2 after 22 min

$\Rightarrow$ Doubles after 22 min

$\Rightarrow$ Doubles after $\frac{22}{60}$ hours

$\frac{11}{30}$ hours

Starting Pop

$= P_0$

$= 200$

Want: $P(t) = P_0 e^{kt}$

$$P\left(\frac{11}{30}\right) = P_0 e^{k \cdot \left(\frac{11}{30}\right)}$$

Solve for k (unknown):

$$2 \cdot 200 = 200 \cdot e^{k\left(\frac{11}{30}\right)}$$

$$2 = e^{k\left(\frac{11}{30}\right)}$$

$$\ln(2) = k \cdot \left(\frac{11}{30}\right)$$

$$k = \frac{\ln(2) \cdot 30}{11}$$

(a) $P(t) = P_0 \cdot e^{kt}$

$$\boxed{P(t) = 200 \cdot e^{\left(\frac{\ln(2) \cdot 30}{11}\right)t}}$$

(b) $P(10) = 200 \cdot e^{\left(\frac{\ln(2)\cdot 30}{11}\right)\cdot 10}$

$= 200 \cdot e^{\left(\ln(2)\cdot \frac{300}{11}\right)}$

$= 200 \cdot \left(e^{\ln(2)}\right)^{\frac{300}{11}}$

$= 200 \cdot (2)^{\frac{300}{11}}$

(c) $10{,}000 = 200 \cdot e^{\left(\frac{\ln(2)\cdot 30}{11}\right)\cdot t}$

$200\overline{)10{,}000}$ 50

$50 = e^{\frac{\ln(2)\cdot 30}{11} t}$

$\ln 50 = \frac{\ln(2)\cdot 30}{11} \cdot t$

$\frac{\ln(50)}{\ln(2)} \cdot \frac{11}{30} = t \approx 2.1$ hours.

(3) 500 at 1pm (0 hrs after 1pm)
4000 at 6pm (5 hrs after 1pm)

$$P(t) = P_0 e^{kt}$$

$$P(5) = 500 e^{k \cdot 5}$$
$$4000 = 500 e^{k \cdot 5}$$
$$500 \overline{)4000}^{\,8} \qquad 8 = e^{5k}$$

$$\ln(8) = 5k$$
$$k = \frac{\ln(8)}{5}$$

(a) $$P(t) = P_0 e^{kt}$$
$$P(t) = 500 e^{\frac{\ln(8)}{5} t}$$

(b) Pop at 7pm (6 hrs after 1pm
$$= P(6) = 500 e^{\frac{\ln(8)}{5} \cdot 6}$$

$$= 500 \left(e^{\ln 8}\right)^{\frac{6}{5}}$$

$$= 500 \cdot 8^{\frac{6}{5}}$$

(c) $P(t) = P_0 \, e^{\frac{\ln 8}{5} t}$

$$15000 = 500 \, e^{\frac{\ln 8}{5} t}$$

$$30 = e^{\frac{\ln 8}{5} t}$$

$$\ln(30) = \frac{\ln 8}{5} t$$

$$t = \frac{5 \cdot \ln(30)}{\ln(8)}$$

(d) time to double

$$1000 = 500 \cdot e^{\frac{\ln 8}{5} t}$$

$$2 = e^{\frac{\ln 8}{5} t}$$

$$\ln(2) = \frac{\ln(8)}{5} t$$

$$\frac{5 \cdot \ln(2)}{\ln(8)} = t$$